Playing Quantum Physics Jeopardy with zero-energy eigenstates

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Abstract

We describe an example of an exact, quantitative Jeopardy-type quantum mechanics problem. This problem type is based on the conditions in one-dimensional quantum systems that allow an energy eigenstate for the infinite square well to have zero curvature and zero energy when suitable Dirac delta functions are added. This condition and its solution are not often discussed in quantum mechanics texts and have interesting pedagogical consequences.

I. INTRODUCTION

Students often work backward when problem solving in their introductory physics courses. This method typically entails using the information in the problem and if available, the answer in the back of the book to work backward to determine which equation to use.[1] Students using this novice problem-solving approach have been shown to lack the conceptual foundation of more advanced problem solvers.[2] In more advanced courses such as quantum mechanics, students still apply novice problem-solving approaches and studies have shown that students' conceptual understanding on all levels is lacking.[3]

Van Heuvelen and Maloney[4] have described a problem type that encourages a more conceptual problem-solving strategy called a "working backward" or Jeopardy problem. These problems begin with the answer (an equation, a diagram or a graph, or a simulation) and ask students to work back toward the question, much like the game show Jeopardy.

In Quantum Physics Jeopardy, students are given an energy eigenstate (the answer) and are asked to find the potential energy function (the question) that yields this eigenstate. This problem elucidates the connection between the form of the energy eigenstate and the potential energy function in one-dimensional quantum systems.[5] The technique is similar to inverse problems in quarkonium spectroscopy where constraints on the binding potential from the bound-state energies are obtained.[6] Applications of inverse methods to other fields, such as medical imaging[7,8] (for example, CT scans) and geophysics[9] are even more familiar.

This paper describes a new class of quantitative Quantum Physics Jeopardy problems that exploit a zero-curvature ($E = 0$) energy eigenstate that occurs when suitably chosen Dirac delta functions are added to an infinite well. These special configurations are often overlooked,[10] but they provide additional exactly-solvable problems in quantum mechanics.

II. INFINITE WELL WITH DELTA FUNCTIONS

In the infinite square well the $E = 0$ energy eigenstate is rejected by most textbooks on the grounds that the general form for energy eigenstates, $A\sin(kx + \phi)$, does not yield a physical solution when $k = 0$. As pointed out in Ref. 11 the $E = 0$ energy eigenstate of the Schrödinger equation is not a sinusoidal function. Instead, $E - V(x)$ is zero inside the well and the time-independent Schrödinger equation simplifies to $d^2\psi(x)/dx^2 = 0$, and yields

$\psi(x) = Ax + B$, where A and B are constants. In this context the energy eigenstate cannot be normalized and still satisfy the boundary and continuity conditions for the infinite square well, and thus the $E = 0$ state cannot be an allowed energy eigenstate.

Although the authors of Ref. 11 used the zero-curvature case to show that the infinite square well cannot have a zero-energy eigenstate, cases in which zero-curvature states are valid are seldom considered,[12,13] even though, for example, all of the energy eigenstates for the infinite square well are linear (namely zero) outside of the well.

A zero-energy eigenstate can occur with the addition of a single attractive Dirac delta function , $V_1(x) = -\alpha\delta(x)$ with $\alpha > 0$, at the origin of a symmetric well with infinite walls at $x = \pm L$. We write the delta function in this way to work with positive constants and to make explicit the minus sign in $V_1(x)$. Similar scenarios have been considered[14,15] and are related to experiments in which a potential energy "spike" inserted in a quantum well is modeled by a Dirac delta function.[16,17] Our approach differs in that we tune α to be $\frac{2}{L}(\frac{\hbar^2}{2m})$ so that an energy eigenstate of zero energy arises. The additional potential energy function splits the well into two regions: region I ($-L < x < 0$) and region II ($0 < x < L$). Assuming a zero-energy eigenstate exists, continuity requires that it be represented as $\psi_{\mathrm{I}}(x) = A(x+L)$, $\psi_{\mathrm{II}}(x) = -A(x - L)$, and zero outside the well. We then ensure that the energy eigenstate has the proper discontinuity in its slope at the origin due to the Dirac delta function. In general, when the Dirac delta function occurs at the position x_0, we must have that

$$\psi'(x_{0+}) - \psi'(x_{0-}) = -\alpha(2m/\hbar^2)\psi(x_0). \tag{1}$$

It is easy to show that the energy eigenstate and the chosen value of α satisfy Eq. (1). Figure 1 shows the eigenstate corresponding to a symmetric infinite square well with $V_1(x)$ added, $\psi_1(x)$, which is also a limiting case of an analysis that used supersymmetric quantum mechanics.[15] There are an indefinite number of combinations of Dirac delta functions that when added to the infinite square well result in zero-energy eigenstates.

We can also proceed in the opposite direction as in Jeopardy: write any piecewise linear (single-valued) energy eigenstate that vanishes at $\pm L$ and does *not* vanish at a kink and determine the Dirac delta function potential(s) that must be added to the infinite square well. Consider the M-shaped, zero-energy eigenstate $\psi_2(x)$ in Fig. 1 and determine V_2. A quantitative result is possible from direct measurement of the energy eigenstate slopes at the kinks and their positions. Because Eq. (1) is independent of an overall multiplicative factor

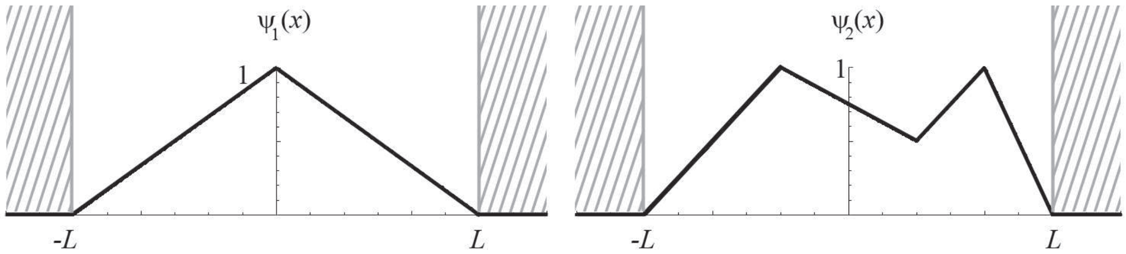

FIG. 1: Two unnormalized zero-energy eigenstates corresponding to two different sets of Dirac delta function(s) added to a symmetric infinite square well.

in $\psi(x)$, we can even begin with an unnormalized energy eigenstate. The answer turns out to be $V_2 = -\frac{9}{4L}(\frac{\hbar^2}{2m})[\delta(x+L/3) - 2\delta(x-L/3) + 2\delta(x-2L/3)]$.

We have created a worksheet template for these Jeopardy exercises.[18,19] Students can be given individual problems by replacing the figure in the worksheet with another drawing of a piecewise linear and single-valued energy eigenstate that vanishes at the infinite walls and does not vanish at a kink; the eigenstate can even be drawn with a ruler and graph paper.

A positive side effect of these Jeopardy problems is that they further illustrate how energy eigenstates with obvious kinks can be valid states. This fact is not discussed in most introductory texts on quantum mechanics. Energy eigenstates must be smooth only if their corresponding potential energy function is well behaved. Only infinite walls (such as in the boundaries of the infinite square well) and Dirac delta functions behave badly enough to generate kinks in energy eigenstates. It can also be shown that these states exhibit kinetic/potential energy sharing, by calculating $\langle \hat{T} \rangle$ and $\langle V_\delta \rangle$ which yields an exact cancellation so that $\langle E \rangle = 0$ for these states (as expected). This kinetic/potential analysis provides another illustration of the quantum-mechanical virial theorem.[13]

III. CONCLUSION

Zero-energy eigenstates extend the standard treatment of the infinite square well and other piecewise-constant potential energy wells.[20] Although these states seem like an intuitively natural interpolation between the much more commonly discussed oscillatory and tunneling solutions, the unfamiliar mathematical form of the one-dimensional Schrödinger equation for situations where $E - V(x)$ is zero over an extended region of space catches many students by surprise.[12]

Acknowledgments

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[18] See `<http://webphysics.davidson.edu/mjb/jeopardy/>`.

[19] The accompanying worksheets are deposited at EPAPS Document No. xxx. This document may be retrieved via the EPAPS homepage (`<http://www.aip.org/pubservs/epaps.html>`) or from `ftp.aip.org` in the `directory/epaps/`. See the EPAPS homepage for more information.

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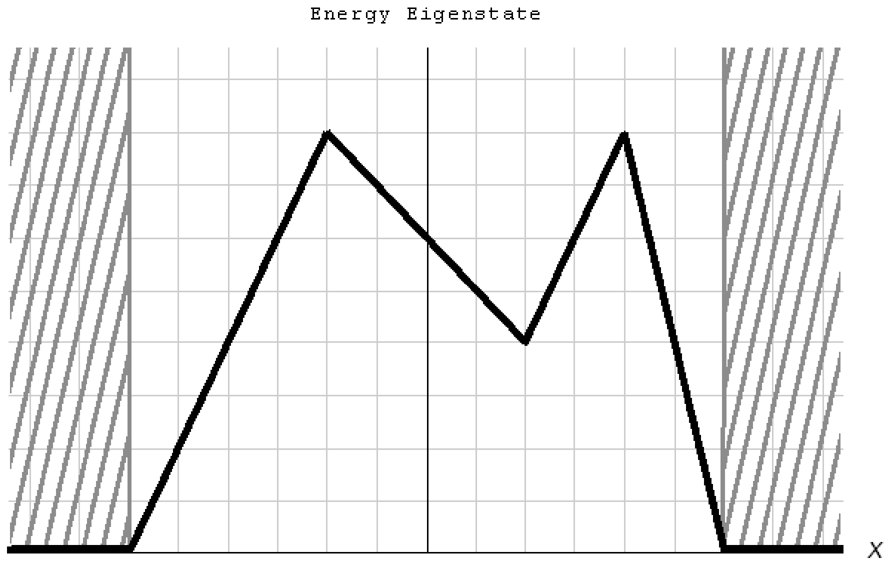
Quantum Physics Jeopardy Problem



This exercise requires you to work backward from the answer (the energy eigenstate) to the question (the potential energy function). Such working-backward exercises are called Physics Jeopardy problems as they are similar to the game show *Jeopardy*.

Shown is an $E = 0$ energy eigenstate for a particle of mass m confined by infinite walls at $x = -L$ and $x = L$. Note that the energy eigenstate is linear in four segments. Also note that the slope of the energy eigenstate changes abruptly at $x = -L/3$, $L/3$, and $2L/3$, creating kinks in the state. Answer the questions below in terms of m, L, and $\hbar$.

1. Determine the solution to the time-independent Schrödinger equation for regions of space where $V(x)$ is a constant, V_0, and $E = V_0$.

2. Determine the mathematical form of the energy eigenstate from the diagram. *Hint*: determine the equation of each line that makes up the energy eigenstate and then piece together.

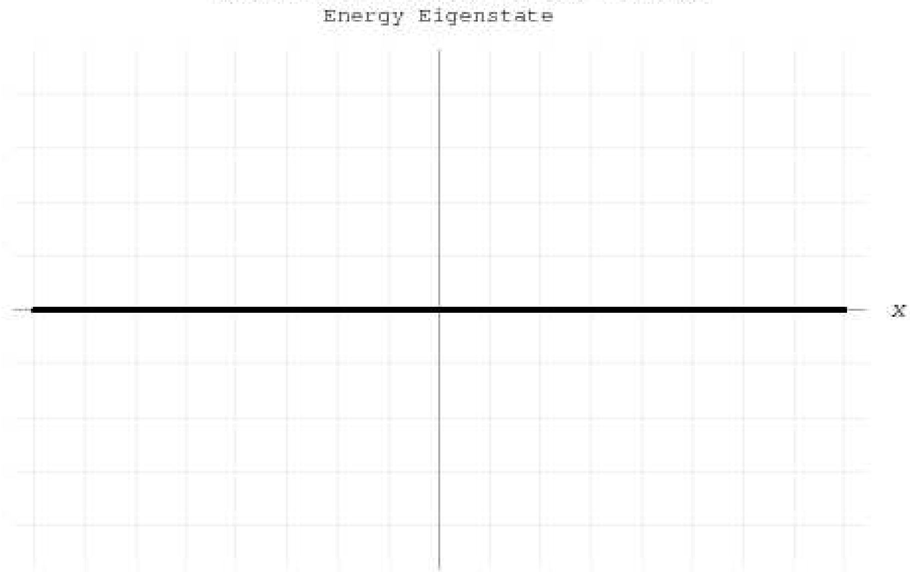
3. Is the energy eigenstate normalized? If not, normalize.

4. What potential energy function(s) need to be added to the well to yield this energy eigenstate?

5. Does your answer to Question 4 dependent on whether the energy eigenstate is normalized? Why or why not?

Name:______________

Quantum Physics Jeopardy Problem



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Shown is an $E = 0$ energy eigenstate for a particle of mass m confined by infinite walls at $x = -L$ and $x = L$. Note that the energy eigenstate is linear in segments. Also note that the slope of the energy eigenstate changes abruptly (it has kinks). Answer the questions below in terms of m, L, and $\hbar$.

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